

Four-level systems and a universal quantum gate

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May 16, 2008

Abstract

We discuss the possibility of implementing a universal quantum XOR gate by using two coupled quantum dots subject to external magnetic fields that are parallel and slightly different. We consider this system in two different field configurations. In the first case, parallel external fields with the intensity difference at each spin being proportional to the time-dependent interaction between the spins. A general exact solution describing this system is presented and analyzed to adjust field parameters. Then we consider parallel fields with intensity difference at each spin being constant and the interaction between the spins switching on and off adiabatically. In both cases we adjust characteristics of the external fields (their intensities and duration) in order to have the parallel pulse adequate for constructing the XOR gate. In order to provide a complete theoretical description of all the cases, we derive relations between the spin interaction, the inter-dot distance, and the external field.

1 Introduction

At present there exists a belief that single quantum systems, e.g., atomic traps, and electronic devices whose operation involves only a few number of electrons, could be used to implement the so-called quantum gates for quantum computations. Quantum computations (quantum computers, or simply QC in what follows) can efficiently solve problems that are considered intractable by classical computers, e.g., the factoring of primes [1] and the simulation of others quantum systems [2]. As for classical computers, for the QC the accomplishment of an arbitrary algorithm can be performed using just a few specific manipulations called universal quantum gates. With these universal quantum gates, a process that acts in an arbitrary number of quantum bits (qubits) can be constructed using gates that act only in one and two qubits.

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Although the first realizations of quantum computers were performed using trapped ions [3], NMR [4], and optical cavities [5], it is believed that, as for classical computers, for QC the most promising object for a possible large-scale implementation, i.e., a implementation involving a large number of qubits, will be the solid-state devices. Among these devices, one can highlight the system of two coupled semiconductor quantum dots (QD) [6]. In this system the qubits are the one-half spin states of an excess electron in each dot, and the universal quantum gates can be obtained by performing arbitrary rotation of the individual spin (one-qubit gate) and one operation capable of entangling the spins of the two electrons (two-qubits gate). Any such operation with this entangling feature can be used [7]. The rotations and the entanglement operations can be performed by applying external electromagnetic pulses to the dots. These pulses can be applied in a serial manner, which perform sequentially the one and the two-qubit gates using a sequence of pulses, or in a parallel manner, where the two and one-qubit gates are performed at once by the application of just one adequate pulse (called a *parallel pulse*). Due to the decohering effects it is important to perform the operations in a minimal time, which, as discussed in [8], can be achieved more adequately with a parallel pulse.

The adequate parallel pulse to implement a quantum gate can be obtained if we know the Hamiltonian that describes the coupled QD, but to construct this Hamiltonian, one has to know the exact relation between the external parameters of the system (such as the external fields and the physical characteristics of the dot) and the interaction between the electrons. An advance in the description of this relation was obtained in [9], where the authors have used the Heitler-London approximation to obtain an explicit expression that relates the interaction between the dots, the inter-dots distance and the external electromagnetic field. However, in this article, the two dots were subject to the same magnetic field, and, as we will see, a little difference between the fields in the dots possess a great influence in the evolution of the system and, consequently, in the implementation of the gates. In addition, after the Hamiltonian is known, we have to construct exact solutions of the evolution equation of this four-level system. In our work [10] we studied possible exact solutions of a system of two coupled spin one-half particles subject to external time-dependent magnetic fields.

In the present work we discuss one possibility of implementing an universal quantum gate, namely the *exclusive or* (XOR) gate, by using two coupled QD subjected to external magnetic fields that are parallel and slightly different. First, in Sections 2 and 3, we make a brief discussion of the model with the parallel external magnetic fields of general form, and describe possible corresponding exact solutions from our previous work [10]. In order to have a complete theoretical description of the problem, one needs to find relations between the spin interaction and external fields. This problem is solved in Section 4. To this end we generalize results of [9], derived for equal parallel external fields, to the case of different fields, obtaining relations between the spin interaction, the inter-dot distance, and the external field. In section 5 we consider two parallel fields with specific form. In the first case, parallel external fields with intensity difference

at each spin being proportional to the interaction between the spins, the latter depending on time. General exact solution describing this system is presented and analyzed to adjust field parameters. Then we consider parallel fields with intensity difference at each spin being constant and interaction between the spins switching on and off adiabatically. In both cases we adjust characteristics of the external fields (their intensities and duration) in order to have the parallel pulse adequate for constructing the XOR gate.

2 Four-level system

In non relativistic quantum mechanics the dynamics of a fixed spin one-half particle, subject to a time-dependent external field $\mathbf{K}(t)$, can be described by the Hamiltonian $\hat{h} = (\boldsymbol{\sigma} \cdot \mathbf{K})$ [11], where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and we set $\hbar = 1$. The quantum Hamiltonian of two interacting one-half spins subjected to external fields \mathbf{G} and \mathbf{F} , respectively, is chosen as, see e.g. [12],

$$\hat{H}(\mathbf{G}, \mathbf{F}, J) = (\boldsymbol{\rho} \cdot \mathbf{G}) + (\boldsymbol{\Sigma} \cdot \mathbf{F}) + \frac{J}{2} (\boldsymbol{\Sigma} \cdot \boldsymbol{\rho}), \quad (1)$$

where the function $J = J(t)$ defines the spin interaction, $\mathbf{G} = (G_1(t), G_2(t), G_3(t))$ and $\mathbf{F} = (F_1(t), F_2(t), F_3(t))$ are the time-dependent external fields at each particle, and the 4×4 Dirac-matrices $\boldsymbol{\rho}$ and $\boldsymbol{\Sigma}$ are in the standard representation

$$\boldsymbol{\Sigma} = I \otimes \boldsymbol{\sigma}, \quad \boldsymbol{\rho} = \boldsymbol{\sigma} \otimes I, \quad (\boldsymbol{\Sigma} \cdot \boldsymbol{\rho}) = \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} = \sum_{i=1}^3 \sigma_i \otimes \sigma_i, \quad (2)$$

where I is the 2×2 identity.

The interaction of the spins in (1) is known as the *Heisenberg interaction* and describes, for example, the interactions in the well known Hubbard model [13]. In particular, under certain conditions, this interaction can be used to describe the coupling between two QD [6]. In this case, it is possible to control not only the external fields, but also the spin interaction. When these QD are used to implement a quantum gate, the operation of the gate is performed by varying the external fields and the spin interaction during a certain time τ . To this end, in order to justify the Heisenberg interaction, the following conditions (see [6]) must hold:

1. The time τ cannot be too small to avoid transitions to higher energy levels, so that the time scale should be bigger than $\hbar/\Delta E$, where ΔE is the difference between the first two energy levels;
2. The decoherence time of the physical system should be much bigger than τ .

The use of these QD to implement quantum algorithms requires controlling the individual spins, which can be done by the fields \mathbf{F} and \mathbf{G} , and any interaction between the spins capable of creating an entangled state starting from a

original product state [7]. A system described by the Hamiltonian (1) satisfies these conditions. Namely, when the fields \mathbf{F} and \mathbf{G} are zero, the evolution operation $R_t(\mathbf{G}, \mathbf{F}, J)$ of the Hamiltonian (1) can be written as [10]

$$R_t(0, 0, J) = \exp[i\Phi(t)/2] [\mathbb{I} \cos \Phi(t) - iA \sin \Phi(t)] ,$$

$$A = \frac{1}{2} [\mathbb{I} + (\boldsymbol{\Sigma} \cdot \boldsymbol{\rho})] , \quad \Phi(t) = \int_{t_0}^t J(\tau) d\tau , \quad (3)$$

where \mathbb{I} is the 4×4 unity matrix. From the above expression we see that, when $\Phi = \pi/4$, the evolution operator acts as the universal gate known as square root of swap ($U_{\text{sw}}^{1/2}$). The interaction function J can experimentally be controlled in many different ways, e.g., by applying electrical or magnetic fields to the dots [14]. So we can construct any quantum gate by a sequence of pulses by turning on and off the external fields and the interaction. For example, the XOR gate can be constructed as [9]

$$U_{\text{XOR}} = \exp\left(i\frac{\pi}{4}\rho_3\right) \exp\left(-i\frac{\pi}{4}\Sigma_3\right) U_{\text{sw}}^{1/2} \exp\left(i\frac{\pi}{2}\rho_3\right) U_{\text{sw}}^{1/2} . \quad (4)$$

On the other hand, the construction of gates by a sequence of pulses is not appropriate, because the duration of all the sequence can be too long, violating condition (2) above, or the pulses need to vary too fast, violating the condition (1). So it is important to be able to implement the gates at once, applying just one adequate field. This single pulse is called parallel pulse [8]. For example, in the case of the XOR gate (4), to use a parallel pulse one needs to find a field whose evolution operation, at a given instant τ of time, has the form

$$R_\tau(\mathbf{G}, \mathbf{F}, J) = U_{\text{XOR}} = \exp\left[-i\frac{\pi}{4}(\Sigma_3\rho_3 + \Sigma_3 + \rho_3)\right] . \quad (5)$$

In order to find this parallel pulse, in a general case, we need to know exact solutions of the Schrödinger equation with the Hamiltonian (1) for different kinds of external fields and interactions. A large number of these exact solutions are present in our previous work [10], and here we will use some of these results to describe the implementation of a U_{XOR} gate. The procedure developed here can easily be extended to others gates using others fields present in [10].

3 Parallel external fields

Let each spin in our system be subject to different time-dependent external magnetic fields along the z direction,

$$\mathbf{G} = (0, 0, \mu_B g_1 B_1) , \quad \mathbf{F} = (0, 0, \mu_B g_2 B_2) , \quad B_{1,2} = B_{1,2}(t) . \quad (6)$$

where μ_B is the Bohr magneton and g_i the g -factor of the dot i . Therefore the Hamiltonian (1) assumes the form

$$\hat{H} = \frac{1}{2} [(\Sigma_3 + \rho_3) B_+ - (\Sigma_3 - \rho_3) B_- - J] + AJ , \quad B_\pm = \mu_B (g_1 B_1 \pm g_2 B_2) , \quad (7)$$

with the constant 4×4 orthogonal matrix A given in (3). For the first and fourth components v_1 and v_4 of the four-spinor Ψ , solution of the Schrödinger equation ($i\dot{\Psi} = \hat{H}\Psi$) with the above Hamiltonian, we get

$$v_1 = C_1 \exp \left[-i \int_0^t \left(\frac{J}{2} + B_+ \right) d\tau \right], \quad v_4 = C_4 \exp \left[-i \int_0^t \left(\frac{J}{2} - B_+ \right) d\tau \right], \quad (8)$$

where $C_{1,4}$ are complex constants. Besides, for the components $v_{2,3}$ of Ψ , we obtain the expression

$$i\dot{\psi}' = \left[(\boldsymbol{\sigma} \cdot \mathbf{K}) - \frac{J}{2} \right] \psi', \quad \psi' = \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}, \quad (9)$$

$$\mathbf{K}(t) = (J(t), 0, B_-(t)). \quad (10)$$

Making the transformation

$$\psi'(t) = \exp \left[\frac{i}{2} \int_0^t J(\tau) d\tau \right] \psi(t), \quad (11)$$

the two-component spinor ψ obeys the equation for a single particle with spin one-half in an effective external field $\mathbf{K}(t)$, i.e., $i\dot{\psi} = (\boldsymbol{\sigma} \cdot \mathbf{K}) \psi$. Consequently, in this case, the four-level system problem reduces to finding solutions of a two-level system. Writing the solution of this two-level-system as $\psi(t) = \hat{u}_t \psi(0)$, using the 2×2 evolution operator $\hat{u}_t(J, B_-)$, we can write the evolution operator of the four-level-system governed by the Hamiltonian (7) as

$$R_t(\mathbf{G}, \mathbf{F}, J) = \exp \left(-\frac{i}{2} [(\Sigma_3 + \rho_3) \Gamma(t) + \Sigma_3 \rho_3 \Phi(t)] \right) M(t), \quad (12)$$

$$\Gamma(t) = \int_0^t B_+(\tau) d\tau, \quad \Phi(t) = \int_0^t J(t) d\tau,$$

with the 4×4 matrix $M(t)$ given by components of the single particle evolution operator \hat{u}_t as

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \hat{u}_t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Writing the effective field \mathbf{K} (10) as

$$\mathbf{K} = \frac{\dot{\mathbf{q}} + [\mathbf{q} \times \dot{\mathbf{q}}]}{1 + \mathbf{q}^2}, \quad \dot{q}_2 = q_1 \dot{q}_3 - q_3 \dot{q}_1, \quad (13)$$

for an appropriate real three-vector $\mathbf{q}(t)$, we can obtain the evolution operator \hat{u}_t in the general form [11]

$$\hat{u}_t = \frac{1 + \mathbf{q}\mathbf{q}_0 - i\sigma\mathbf{p}}{\sqrt{(1 + \mathbf{q}^2)(1 + \mathbf{q}_0^2)}}, \quad \mathbf{p} = \mathbf{q} - \mathbf{q}_0 + [\mathbf{q}_0 \times \mathbf{q}], \quad (14)$$

where $\mathbf{q}_0 = \mathbf{q}(0)$. Thus we can construct the XOR operator (5) by using any effective field \mathbf{K} given by a vector \mathbf{q} that, after a certain time T , points again in the initial direction ($\mathbf{q}(T) \propto \mathbf{q}_0$). When this is the case, $\hat{u}_T = I$ and the evolution operator R_t (12) will assume the form U_{XOR} (5) when

$$\Gamma(T) = \Phi(T) = \frac{\pi}{2} \bmod(2\pi) .$$

This periodicity in the direction of the vector \mathbf{q} can be considered a general property for which the parallel field acts as a XOR gate. An arbitrary choice of this vector, respecting only this periodic condition, can be used to obtain a variety of parallel fields that will certainly possess the adequate characteristics.

Although the interaction function J depends on the applied external fields, as we will see in the next section, when $B_- \ll B_+$, we can make $J = J(B_+)$ and consider the vector \mathbf{K} in (10) as composed of two independent functions $J(t)$ and $B_-(t)$. Besides, the interaction function can be controlled by electric fields [14], whose interference in the spin states via the spin-orbit coupling, in many practical application, can be neglected. Then, if necessary, we can consider the functions J , B_- and B_+ independently. In addition, although the obtained expressions depend on the sum and on the difference between the fields at the spins, only the average value of this sum is relevant, so that the explicit form of its time variation can be arbitrary.

Some comments on the experimental realization are in order. With the technology available nowadays, systems of two coupled QD, each holding exactly one electron, are routinely constructed in experimental solid state physics [15]. The confinement of the electron and the control of the interaction between them can be performed by electrical gates directly assembled in the semiconductor material using etching techniques. The coupled electron spin is a result of the combination of the Coulomb interaction and the Pauli Exclusion Principle. The spin state of this system can be initialized using a variety of techniques [16, 14]. In all experiments with such systems, the difference between the energy of singlet and triplet spin states are controlled by applying either static or time-dependent external magnetic fields of intensity up to the order of a few teslas. See, for example, the experimental setup [14]. Therefore, the construction of systems of two-coupled QD subject to external parallel magnetic fields is commonplace. However, the main problem in the experimental application of the results presented in this work is to generate, and control, a difference in the magnetic field felt by each spin, that is, the manipulation of B_- (7). The quantity B_- is not necessarily associated with a gradient in the applied magnetic field. It is possible to couple two different QD [17] in such a way that $g_1 \neq g_2$. In addition, some techniques permit the manipulation of the g -factor by changing the size of the dots or by the application of external electromagnetic fields [18, 19]. The hyperfine coupling between the electron spin and the nuclear spins of the semiconductor material can also be explored to obtain a field gradient by producing a differential Overhauser field [20]. As a result, we can have a nonzero B_- even when the two dots are subject to the same external magnetic field. Besides, it is possible to fine-tune the quantity B_- with the

application of a localized magnetic field [21]. In [22] an experimental setup of this kind is used to control the spin state of a single electron in a coupled QD system. The fine-tuning of the quantity B_- has been an intense object of study in the current solid state experiments.

4 Relation between the interaction function and the external fields

In order to construct the evolution operator (12) for some specific parallel field, one needs to know the explicit dependence between the external fields and the interaction function. In the article [9], the authors study two QD, coupled with the Heisenberg interaction (1), and use the Heitler-London approximation to obtain an expression for the interaction function $J(V, a, B, E)$ as a function of the gate voltage between the coupled QD (V), the inter-dot distance (a), an external magnetic field (B) and an external electric field (E). The knowledge of the interaction as a function of various parameters allows to compensate the changes in J with the variation of one given parameter by controlling another parameter. So one can, for example, vary the magnetic field B and maintain J constant by changing the inter-dot potential V . This procedure allows treating some parameters of the Hamiltonian of the system as independent. However, the analysis developed in [9] concerns only the problem of two dots subject to the same magnetic field. Since the work developed here depends on the difference between the fields at each dot, we will repeat the process given in the cited article, but with different magnetic fields at each dot. For different parallel magnetic fields (B_1, B_2) applied at the spins, the non-homogeneity of the field make it very complicated to calculate $J(B_1, B_2)$, since, in this case, the Zeeman terms ($\boldsymbol{\rho} \cdot \mathbf{G}$ and $\boldsymbol{\Sigma} \cdot \mathbf{F}$) will not be independent of the space coordinates anymore. Despite this fact, since only a small field difference is important, the fields can be considered homogeneous inside each dot and the Heitler-London approximation can still be used to describe the interaction function.

The model consists of two identical bi-dimensional dots with an inter-dot separation of $2a$, each dot with one electron of charge e and subject, respectively, to the magnetic field B_1 and B_2 in the z direction. The field B_i is considered homogeneous inside the dot i . In the Heitler-London approximation, we start with a combination of the orbital ground states of bi-dimensional single dots and combine this state in symmetric ($|\Psi_+\rangle$) and anti-symmetric ($|\Psi_-\rangle$) states for the double dot problem as

$$|\Psi_{\pm}\rangle = \frac{|-+\rangle \pm |+-\rangle}{\sqrt{2 \pm S^2}}$$

where $\varphi_{\pm}(\mathbf{r}) = \langle \mathbf{r} | \pm \rangle$ denotes the one-particle orbital centered at $\mathbf{r} = (\pm a, 0)$ and $S = \langle + | - \rangle$ the overlap between the two orbitals. The interaction function, or the exchange energy, is the difference between the energies, $J = \langle \Psi_- | H_{orb} | \Psi_- \rangle -$

$\langle \Psi_+ | H_{orb} | \Psi_+ \rangle$ where the double-dot orbital Hamiltonian is given by

$$\begin{aligned} H_{orb} &= h^0(\mathbf{r}_1) + h^0(\mathbf{r}_2) + C, \quad C = \frac{e^2}{\kappa |\mathbf{r}_1 - \mathbf{r}_2|}, \\ h^0(\mathbf{r}_i) &= \frac{1}{2m} \left[\mathbf{p}_i - \frac{e}{c} \mathbf{A}(\mathbf{r}_i) \right]^2 + V(\mathbf{r}_i), \quad i = 1, 2, \\ \mathbf{A}(\mathbf{r}_i) &= \frac{B_i}{2} (-y_i, x_i, 0), \quad V(x, y) = \frac{m\omega_0^2}{2} \left[\frac{1}{4a^2} (x^2 - a^2)^2 + y^2 \right]. \end{aligned} \quad (15)$$

In this expression c is the speed of the light, κ is the dielectric constant of the medium, m is the effective mass of the electron (e.g., $m = 0.067m_e$ in GaAs), \mathbf{r}_i and \mathbf{p}_i are the position and the momentum of the i -th electron, B_i is the magnetic field at dot i and the harmonic potential well $V(\mathbf{r})$ of frequency ω_0 is motivated by experimental results [23]. The matrix elements needed to calculate J can be obtained by adding and subtracting in (15) the harmonic potentials V_{\pm} centered at $x_i = (-1)^i a$ for the i -th electron,

$$\begin{aligned} H_{orb} &= h_-^0(\mathbf{r}_1) + h_+^0(\mathbf{r}_2) + W + C, \\ h_{\pm}^0(\mathbf{r}_i) &= \frac{1}{2m} \left[\mathbf{p}_i - \frac{e}{c} \mathbf{A}(\mathbf{r}_i) \right]^2 + V_{\pm}(\mathbf{r}_i), \\ V_{\pm}(\mathbf{r}_i) &= \frac{m\omega_0^2}{2} \left[(x_i \pm a)^2 + y_i^2 \right], \quad W = \sum_{i=1}^2 V(\mathbf{r}_i) - [V_-(\mathbf{r}_1) + V_+(\mathbf{r}_2)], \end{aligned} \quad (16)$$

and using the ground states functions φ_{\pm} centered at $\mathbf{r} = (\pm a, 0)$,

$$\begin{aligned} \varphi_{\pm}(\mathbf{r}) &= \sqrt{\frac{m\omega_{\pm}}{\pi\hbar}} \exp\left(\pm \frac{im\omega_{L\pm}}{\hbar} y\right) \exp\left(-\frac{m\omega_{\pm}[(x \mp a)^2 + y^2]}{2\hbar}\right), \\ \omega_{\pm} &= \sqrt{\omega_0^2 + \omega_{L\pm}^2}, \quad \omega_{L-(+)} = eB_{1(2)}/2mc, \end{aligned} \quad (17)$$

which are eigenfunctions of h_{\pm}^0 with energy ω_{\pm} ,

$$h_{-(+)}^0(\mathbf{r}_{1(2)}) \varphi_{-(+)}(\mathbf{r}_{1(2)}) = \omega_{-(+)} \varphi_{-(+)}(\mathbf{r}_{1(2)}).$$

With this, the expression for the interaction function becomes

$$\begin{aligned} J &= \frac{2S^2}{(1-S^4)} \left[L - \frac{\hbar\omega_0}{4} \frac{(b_+^2 - b_-^2)(b_- - b_+)}{b_+ b_-} \right], \\ L &= \left(\langle 12 | C + W | 12 \rangle - \frac{\text{Re} \langle 12 | C + W | 21 \rangle}{S^2} \right), \quad b_{\pm} = \omega_{\pm}/\omega_0, \end{aligned} \quad (18)$$

using the expression (17) for the functions $\varphi_{\pm}(\mathbf{r}_i) = \langle \mathbf{r}_i | \pm \rangle$ we get

$$\begin{aligned} \frac{S^2}{(1-S^4)} &= \frac{1 - \Delta^2}{(2 \sinh(2M) + \Delta \exp(-2M)(2 - \Delta^3))}, \\ M &= \frac{2d^2}{b_+ + b_-} \left[b_- b_+ + \frac{(\omega_{L+} + \omega_{L-})^2}{4\omega_0^2} \right], \quad \Delta = \frac{b_- - b_+}{b_- + b_+}, \quad d = \frac{a}{a_0}, \end{aligned}$$

where $a_0 = \sqrt{\hbar/m\omega_0}$ is the effective Bohr radius of the dot. For the matrix elements of W in (18) we have

$$\langle 12|W|12\rangle - \frac{\langle 12|W|21\rangle}{S^2} = \frac{\hbar\omega_0}{2} \left\{ \frac{3}{2d^2(b_- + b_+)^2} \left[\frac{1 + \Delta^2}{(1 - \Delta^2)^2} - 1 \right] - 3 \left(\frac{\Delta^2 - 1}{b_- + b_+} \right) - \frac{d^2}{2} (\Delta^4 - 6\Delta^2 - 3) \right\}$$

and for the matrix elements of the electric interaction C between the electrons

$$\langle 12|C|12\rangle - \frac{\text{Re}\langle 12|C|21\rangle}{S^2} = \frac{e^2}{a_0\kappa} \sqrt{\frac{\pi}{2}} \bar{b} \left\{ \sqrt{(1 - \Delta^2)} \exp[-d^2(1 - \Delta^2)\bar{b}] \text{I}_0[d^2(1 - \Delta^2)\bar{b}] - \exp\left[\frac{d^2}{2}K\right] \text{I}_0\left[\frac{d^2}{2}K\right] \right\},$$

$$K = \bar{b}(1 + \Delta^2) - \frac{1}{\bar{b}} + \sqrt{[(1 - \Delta^2)\bar{b}]^2 - 2(1 + \Delta^2) + \frac{1}{\bar{b}^2}}, \quad \bar{b} = \frac{b_- + b_+}{2},$$

where I_0 is the zeroth-order Bessel function. The quantity Δ is related to the difference and to the sum of the fields ($B'_\pm = B_1 \pm B_2$, without the g -factor) at the dots by

$$\Delta = \frac{B'_+ B'_-}{2 \left[\left(2\hbar \frac{\omega_0}{\mu_B} \right)^2 + B'^2_+ + B'^2_- + \sqrt{\left[\left(2\hbar \frac{\omega_0}{\mu_B} \right)^2 + B'^2_+ + B'^2_- \right]^2 - (2B'_+ B'_-)^2} \right]}.$$

This quantity can be used to evaluate the error in considering J independent of B'_- . In the denominator of the above expression, usually only the first term is relevant, e.g., for a typical GaAs dot we have $\hbar\omega_0/\mu_B = 50$ T. For the same field at both dots we obtain the result given in [9].

5 The exclusive OR gate

Here we establish the parameters of two kinds of parallel external fields that can act as a XOR gate, that is, fields in the form (6) whose evolution operator (12), at a given instant of time, has the form (5). The same analysis presented here can also be carried out for the 26 different families of external fields presented in [11], which gives a wide range of fields to choose from that are more appropriate to adjust to experimental setups. All the development depends on the ability to vary independently the difference B_- and the interaction J , but, as we saw in the previous sections, there are various circumstances when this requirement can be satisfied.

5.1 B_- proportional to the interaction

Suppose that we obtain, theoretically or experimentally, an expression for the interaction function J , and we choose B_- such that it is proportional to J . As

a consequence we can write

$$J(t) = q(t) \sin \lambda, \quad B_-(t) = q(t) \cos \lambda,$$

where $q(t)$ is an arbitrary function of time, and λ is a real constant. Then, the general solution of equation (9) can be written as $\psi(t) = \hat{u}_t \psi(0)$ with the evolution operator \hat{u}_t given by

$$\hat{u}_t = \cos \omega - i(\sigma_1 \sin \lambda + \sigma_3 \cos \lambda) \sin \omega, \quad \omega(t) = \int_0^t q(\tau) d\tau. \quad (19)$$

Selecting the instants T when $\cos \omega = 1$,

$$\omega(T) = \int_0^T q(\tau) d\tau = 2n\pi, \quad n \in \mathbb{N}^*, \quad (20)$$

the evolution operator (12) of this problem assumes the form

$$R_T = \exp(i3n\pi) \exp \left[-\frac{i}{2} (\Gamma [\Sigma_3 + \rho_3] + 2n\pi \Sigma_3 \rho_3 \sin \lambda) \right].$$

So to obtain the XOR gate (5), up to a phase, at every instant T (20), we have to set

$$\Gamma(T) = 2n\pi \sin \lambda = \frac{\pi}{2} \bmod(2\pi),$$

or, in a more explicit form,

$$\sin \lambda = \frac{4m+1}{4n}, \quad m \in \mathbb{N}, \quad m < n.$$

When the interaction J and the difference B_- are constants, so q is a constant, we obtain

$$\Gamma(T) = JT \bmod(2\pi), \quad T = \frac{2n\pi}{\sqrt{J^2 + B_-^2}}. \quad (21)$$

For the special case of a constant sum of fields B_+ , the above expression can be compared with the XOR gate obtained in [8]. To have a idea about the physical values involved in the expression (21), for a typical GaAs QD [9], for an interaction about 50 μeV and a B_- of 10 mT we obtain a XOR gate in a time of 10 ps.

5.2 Adiabatic pulse

The implementation of the quantum gate is performed by the variation of the external fields. However, as described in condition (1) of section 2, these variations can not be so fast so as to prevent the excitation of higher energy-levels. This problem can be avoided by using an adiabatic variation of the fields, which can be obtained by a time-dependence in the form of $\text{sech}(\omega t)$ [8], with $\omega \ll \Delta E/\hbar$

(see condition (1) in section 2). In this section we analyze the case when the field's difference is constant and the interaction varies adiabatically. The case when both the interaction J and the difference B_- vary adiabatically is a special case of the preceding section. The case when the interaction function is maintained constant and the quantity B_- vary adiabatically can be study by knowing that [11]: if ψ is a solution of the equation (9) with the external field $(K_1, 0, K_3)$, then $\tilde{\psi} = (2)^{-1/2} (\sigma_1 + \sigma_3) \psi$ is a solution of (9) with the external field $(K_3, 0, K_1)$. In addition, for B_- not proportional to J a lot of solutions of this sech form can be found in [11].

Suppose that, by vary B_+ or by controlling the electric potential between the dots, we obtain a variation in the form

$$J(t) = a / \cosh \omega t, \quad B_- = c, \quad (22)$$

where a, c are constants. The evolution operator \hat{u}_t of equation (9) with the above fields is given by

$$\hat{u}_t(t) = \frac{1}{|G_2^0|^2 + |G_1^0|^2} \begin{pmatrix} G_1(z) & -G_2^*(z) \\ G_2(z) & G_1^*(z) \end{pmatrix} \begin{pmatrix} G_1^{0*} & G_2^{0*} \\ -G_2^0 & G_1^0 \end{pmatrix}, \quad (23)$$

where the $*$ indicates complex conjugation and

$$\begin{aligned} G_1(z) &= (2c - i\omega) z^{-\nu} (1 - z)^\nu F(\lambda, -\lambda; \gamma; z), \\ G_2(z) &= 2az^{-\nu+1/2} (1 - z)^{\nu+1/2} F(1 + \lambda, 1 - \lambda, \gamma + 1; z), \\ z(t) &= \frac{1}{2} (1 - \tanh \omega t), \quad G_i^0 = G_i(1/2), \\ \nu &= -\frac{ic}{2\omega}, \quad \gamma = 1/2 - 2\nu, \quad \lambda = \frac{|a|}{\omega}. \end{aligned} \quad (24)$$

In these expressions, $F(\alpha, \beta, \gamma, z)$ is the Gauss hypergeometric function.

If we use the sech variation to create a pulse of the interaction function, this pulse will be turned off when $t \gg 1/\omega$. In this time limit we have $\lim_{\omega t \rightarrow \infty} z = e^{-2\omega t}$ and $F(\alpha, \beta, \gamma, 0) = 1$. If in this limit, we choose

$$G_2^0 = aF\left(1 + \lambda, 1 - \lambda, \gamma + 1; \frac{1}{2}\right) = 0, \quad (25)$$

the evolution operator (23) assumes the form

$$\hat{u}_t(\omega t \rightarrow \infty) = \exp(-i\sigma_3 tc). \quad (26)$$

So, following the procedure of the preceding sections, we see that the evolution operator (12) will behave as a XOR gate (5) in the instant T , when $\Gamma(T) = \pi/2 \bmod (2\pi)$ and, for $c \neq 0$,

$$a = \frac{\omega\pi(1 + 4m)}{4 \arctan[\exp(\omega T)] - \pi}, \quad T = \frac{n\pi}{c}, \quad n, m \in \mathbb{N}^*, \quad (27)$$

or, in the limit $\omega T \gg 1$, $a \simeq \omega(1 + 4m)$.

The condition (25) can be obtained in various ways, for example, in the special case when $c = 0$ (and consequently $\lim_{\omega t \rightarrow \infty} \hat{u} = I$ in (26) and we can abandon the condition $T = n\pi/c$) we can use the relation [24],

$$F\left(1 + \lambda, 1 - \lambda; \frac{3}{2}; \frac{1}{2}\right) = \frac{1}{\lambda} \sin\left(\frac{\lambda\pi}{2}\right). \quad (28)$$

So, we obtain the desired condition whenever $|a| = 2m\omega$, $m \in \mathbb{N}^*$. But, once (27) gives a as an odd multiple of ω , it seems that the gate can not be obtained with equal fields at both dots.

6 Some final remarks

We have demonstrated that the general solution of the Schrödinger equation for the system of two interacting spins placed in parallel magnetic fields can be reduced to the general solution of the corresponding equation for only one spin in an effective external field. This allows us to use the known exact solutions of the latter problem to obtain exact solutions for the system of two spins. In turn, this was used by us to describe a construction of a universal quantum gate. We have found restrictions on external magnetic fields (in terms of the effective field for the one-spin system) such that the four level system can act as a XOR gate. The explicit form of the evolution operator and the known dependence of the spin interaction with the magnetic field show that a small difference in the fields applied at each dot (or a small difference in g -factors of these dots) can be explored to control the quantum gate without changing the spin interaction.

One of the problems in the practical application of the theoretical results in constructing universal quantum gates is the knowledge of the spin interaction as a function of external fields. This function can be obtained, in a way unrelated to the Heisenberg interaction model, using a theory that takes into account the physical characteristics of the system, as we do here using the Heitler-London approximation. If from the expression of J , obtained from this more complex model, follows that J does not depend on B_- for $B_- \ll B_+$, we are free to choose B_- as a function of $J(B_+)$. By using this expression of $J(B_+)$, and choosing $B_-(t)$ to be proportional to $J(B_+)$, we can measure the probability transition between the states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ (the swap operation). According to our results, this probability is equal to $\sin^2 \lambda \sin^2 \omega$. This result can be used to test the consistence between the $J(B_+)$, obtained by any model, and the Heisenberg interaction model.

Acknowledgement 1 *M.C.B. thanks FAPESP; D.M.G. thanks FAPESP and CNPq for permanent support.*

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